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HERMITE COLLOCATION METHOD FOR NUMERICAL SOLUTION OF SECOND ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

R. K. NAGAICH[1] & HAPPY KUMAR[2]

[1]Department of Mathematics, Punjabi University, Patiala, Punjab, India

[2]Department of Mathematics, Guru Nanak College, Budhlada (Mansa), Punjab, India

ABSTRACT

A two point boundary value problem has been solved using Hermite collocation method. This technique is a combination of orthogonal collocation and Hermite interpolation polynomial. Zeros of Legendre polynomial has been taken as collocation points. The approximating function has been discretized using cubic Hermite polynomials. The resulting set of equations has been solved using MATLAB ode15s system solver.

KEYWORDS: Hermite Interpolating Polynomials, Orthogonal Collocation, Collocation Points, Boundary Value Problems, Reaction Diffusion Equation

AMS Subject Classification: 35K05, 35K20, 35K57

INTRODUCTION

Solution of two point boundary value problems of the form $L^V(y(x)) = f(x,y)$, where L^V is an operator, has always been a center of interest for the mathematicians around the globe. Numerous methods have been developed such as Spline collocation (Bialecki 1993, Fairweather 1994, Danumjaya & Pani 2005), Finite difference method (Agrawal & Jayaraman 1994), Orthogonal collocation (Soliman & Alhumaizi 1999, Arora et al. 2005), Galerkin method (Cueto et al. 2003), Least square method (Doostan & Laccarino 2009), Homotopy analysis method (Vahdati et al. 2010) etc. to solve two point boundary value problems. Among these above said methods, Finite difference technique and orthogonal collocation technique are the most followed by the investigators in different forms for the solution of different type of models.

Orthogonal collocation on finite elements is combination of weighted residual and variational methods which gives stability as well as accuracy to the results. The Lagrangian interpolation polynomials are widely used to solve two point boundary value problems due to its Kronecker property and easy computability. However, in this interpolating polynomial the trial functions and its first derivative are assumed to be continuous at the node points. It increases the number of collocation equations and thus increases the computational time.

To overcome this property, the orthogonal collocation method is clubbed with Hermite interpolation polynomial. In Hermite collocation method, the approximating function is discretized in terms of cubic Hermite polynomial and then orthogonal collocation is applied within each sub-domain of the global domain. Due to the continuity property of Hermite polynomials there is no need to assume that approximating function and its first derivative should be continuous at node points.

Consider the boundary value problem:

$$\frac{\partial y}{\partial t}=0.25P^{-1}\frac{\partial^2 y}{\partial x^2}-\frac{\partial y}{\partial x}+Ky\text{ ; }(x,t)\in(0,1)\times(0,T) \tag{1}$$

$$K_1 y+K_2\frac{\partial y}{\partial x}=0 \qquad \text{at } x=0 \tag{2}$$

$$K_3 y+K_4\frac{\partial y}{\partial x}=0 \qquad \text{at } x=1 \tag{3}$$

$$y=1 \qquad \text{at } t=0 \tag{4}$$

Where, P, K, K_1, K_2, K_3, K_4 are constants

Hermite Collocation Method

Hermite collocation method is an improvisation of orthogonal collocation method. In this technique, the trial function is approximated by Hermite interpolating polynomials instead of Lagrangian interpolating polynomials. In present study cubic Hermite polynomials have been taken to discretize the approximating function. To apply orthogonal collocation within each element a new variable u is introduced in such a way that $u=\frac{x-x_\ell}{h_\ell}$ where $h_\ell=x_{\ell+1}-x_\ell$ such that u=0 when $x=x_\ell$ and u=1 when $x=x_{\ell+1}$. The interior collocation points are taken to be the roots of the shifted Legendre polynomials.

The Cubic Hermite approximation is defined as

$$y^\ell(u)=\sum_{i=1}^{4}a_i^\ell(t)H_i(u) \text{ Where } \ell=1,2,\ldots\ldots,k \tag{5}$$

Where, k is the number of elements and $a_i^\ell(t)$'s are the continuous functions of 't' in ℓ^{th} element. $\{H_i(u)\}$ are piecewise cubic Hermite polynomials. The cubic Hermite polynomials have also been taken by Carey & Finlayson (1975), Dyksen and Lynch (2000), Brill (2002) etc. to solve the boundary value problems. In present study, the orthogonal collocation is to be defined within each sub-domain, therefore, cubic Hermite polynomials has been defined as:

$$H_1(u)=1-3u^2+2u^3 \tag{6}$$

$$H_2(u)=u^2(3-2u) \tag{7}$$

$$H_3(u)=u(u-1)^2 \tag{8}$$

$$H_4(u)=u^2(u-1) \tag{9}$$

After substituting the interpolating function, the residual obtained is:

$$R^{\ell}(u,t)= \frac{\partial \bar{y}}{\partial t} - 0.25P^{-1}\frac{\partial^2 \bar{y}}{\partial x^2} + \frac{\partial \bar{y}}{\partial x} - K\bar{y} \tag{10}$$

At j[th] collocation point, residual is set equal to zero, i.e.,

$$R^{\ell}(u_j,t)= 0 \qquad \forall\ l = 1, 2,\ldots\ldots k \tag{11}$$

where u_j's are collocation points, since order of each polynomial is three, therefore two interior collocation points are chosen. The first and second order discretized derivatives of the trial function y^{ℓ} taken at j[th] collocation point are defined by A and B, respectively, where $A_{ji}=H'_i(u_j)$ and $B_{ji}=H''_i(u_j)$. After applying Hermite collocation method, the following set of collocation equations is obtained:

$$\frac{dy_j^{\ell}}{dt} = \frac{0.25P^{-1}}{h_{\ell}^2}\sum_{i=1}^{4} a_i^{\ell}(t)B_{ji} - \frac{1}{h_{\ell}}\sum_{i=1}^{4} a_i^{\ell}(t)A_{ji} - Ky_j^{\ell}; \quad \ell=1,2,\ldots,k \tag{12}$$

$$K_1 y_1^1 + \frac{K_2}{h_{\ell}}\sum_{i=1}^{4} a_i^1(t)A_{1i} = 0 \qquad \text{at } x = 0 \tag{13}$$

$$K_3 y_4^k + \frac{K_4}{h_{\ell}}\sum_{i=1}^{4} a_i^k(t)A_{4i} = 0 \qquad \text{at } x = 1 \tag{14}$$

The equations from (12) to (14) can be put into the matrix form as:

$$Ha' = \left(P^{-1}B - A - K\right)a \tag{15}$$

where H is the coefficient matrix of cubic Hermite polynomials at j[th] collocation point. Equation (15) can further be modified as:

$$a' = Qa \tag{16}$$

Where $Q = H^{-1}\left(P^{-1}B - A - K\right)$.

It is clear from the equation (12) to (16) that the resulting system gives 2k number of collocation equations to determine the 2k coefficients.

$$\begin{bmatrix} a_2' \\ a_3' \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1}' \end{bmatrix} = \begin{bmatrix} * & * & * & & & & & & & \\ * & * & * & & & & & & & \\ & * & * & * & * & & & & & \\ & * & * & * & * & & & & & \\ & & & - & - & - & - & & & \\ & & & - & - & - & - & & & \\ & & & & & & & * & * & * \\ & & & & & & & * & * & * \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{bmatrix}$$

Figure 1: Diagrammatic Representation of Equation (16)

The diagrammatic structure of equation (15) is given in Figure 1. It is clear from this figure that $a^{(1)}{}_1$ and $a^{(k)}{}_4$ are determined from boundary conditions.

Error Calculation

The next step is to calculate the error. The relative error for linear problem is calculated by using the formula, $\frac{y_{ex} - y_{nm}}{y_{ex}}$, where y_{ex} is the exact value obtained by conventional method like Laplace transforms (Brenner 1962) and y_{nm} is the numerical value calculated using Hermite collocation method.

Problem

Consider a linear diffusion reaction problem

$$\frac{\partial y}{\partial t} = 0.25 P^{-1} \frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial x} \qquad (x,t)\in(0,1)\times(0,T) \tag{17}$$

$$y - 0.25P^{-1}\frac{\partial y}{\partial x} = 0 \qquad \text{at } x = 0, \text{ for all } t \geq 0 \tag{18}$$

$$\frac{\partial y}{\partial x} = 0 \qquad \text{at } x = 1, \text{ for all } t \geq 0 \tag{19}$$

$$y = 1 \qquad \text{at } t = 0, \text{ for all } x \tag{20}$$

Now, define a trial function as discussed above and after applying cubic Hermite interpolation following system of differential algebraic equations is obtained:

$$\sum_{i=1}^{4} H_{ji} a_i^{'\ell}(t) = \frac{0.25P^{-1}}{h_\ell^2} \sum_{i=1}^{4} B_{ji} a_i^{\ell}(t) - \frac{1}{h_\ell} \sum_{i=1}^{4} A_{ji} a_i^{\ell}(t)\,; \; j = 2, 3 \text{ and } l = 1,2,3\ldots\ldots k \tag{21}$$

$$\sum_{i=1}^{4} H_{1i}(0) a_i^{1}(t) - \frac{0.25P^{-1}}{h_\ell} \sum_{i=1}^{4} H'_{1i}(0) a_i^{1}(t) = 0 \tag{22}$$

$$\sum_{i=1}^{4} H_{4i}'(1) a_i^{k}(t) = 0 \tag{23}$$

In this system using equation (22) and equation (23), the system of differential algebraic equations can be converted into a system of differential equations with $a_1 = 4Pa_2$ and $a_n = 0$

RESULTS AND DISCUSSIONS

The resulting system of 2k equations obtained after discretizing the system of equations is solved using MATLAB with ode15s system solver.

Effect of Parameter P

In Figure 2 the behavior of solution profiles for numerical and analytic values is checked. The numerical and analytic values are compared for P=0.2, 1, 4 and 10. It is observed from this figure that area under the curve increases with the increase in value of P and converges to steady state condition more rapidly for large value of P as compared to small value of P. It is also clear from this figure that analytic and numeric values are matching to a desired limit and numerical values are also converging to steady state condition smoothly.

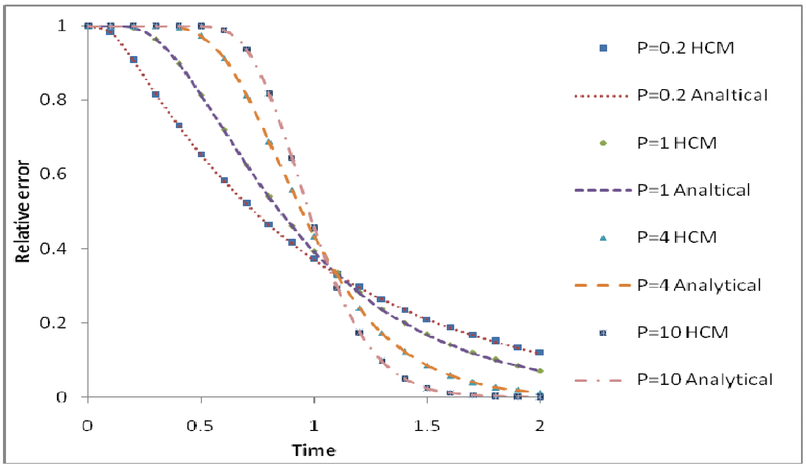

Figure 2: Comparison of Analytic and Numerical Values for Different Values of P

In Figure 3 the behavior of relative error (RE) is shown for P varying from 2 to 10. It is observed that in case of P=2, the relative error is less than 0.02 % which shoots upto 0.08% for P=6 and 0.1% for P=10. It is due to the reason that with the increase in P, the coefficient of $\frac{\partial^2 y}{\partial x^2}$ becomes smaller and equation becomes stiff. Wide variations in the relative error are also observed in Figure 3. It is due to the stiffness of equation, but for large time period too, the relative error is found to be less than 1% for P < 10.

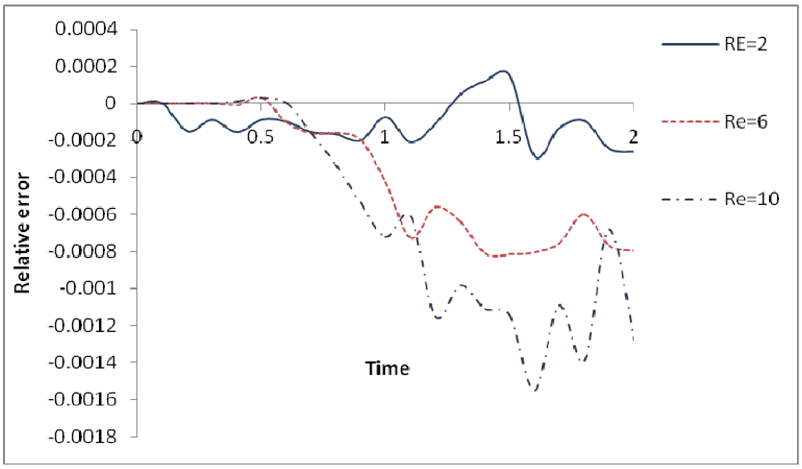

Figure 3: Comparison of Relative Error for Different Values of P

In Figure 4, the behavior of relative error is shown for P varying from 20 to 60 with respect to time. It is clear from this figure that with the increase in the value of P, the coefficient of $\frac{\partial^2 y}{\partial x^2}$ becomes smaller and tends to singularity making equation stiff. This figure shows that in case of P = 20 and P = 40, relative error is less than 0.5% and for P = 60, it is less than 3%.

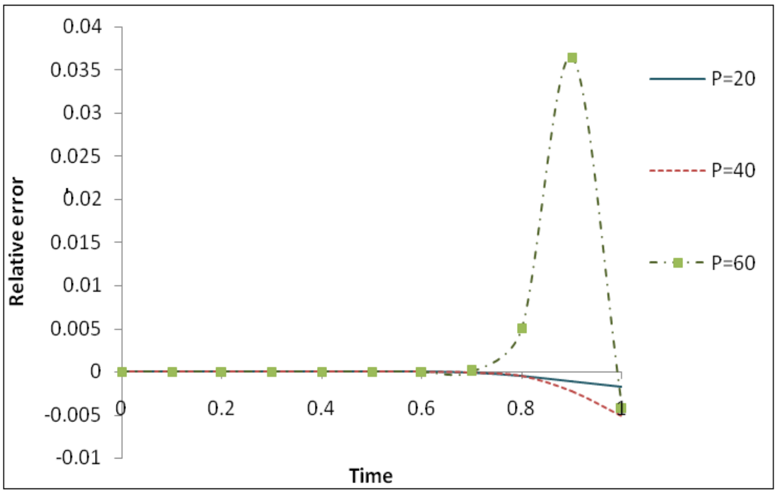

Figure 4: Comparison of Relative Error for Different Values of P

Effect of Number of Elements

In figure 5, the behavior of relative error is shown for different number of elements for P = 40. It is observed from this figure that to reduce the error, the number of elements have gone upto 140. It is due to cubic Hermite interpolation polynomial. In case of cubic polynomial, the number of interior collocation points is two. Due to even number of collocation points and less number, there is wide gap between two collocation points which contribute to increase in error. It can be reduced by increasing the number of elements to span the entire interval [0, 1]. In this figure it is clear that for 35 number of elements, the relative error is upto 2.5% which is less than 1% for 70 and 140 elements.

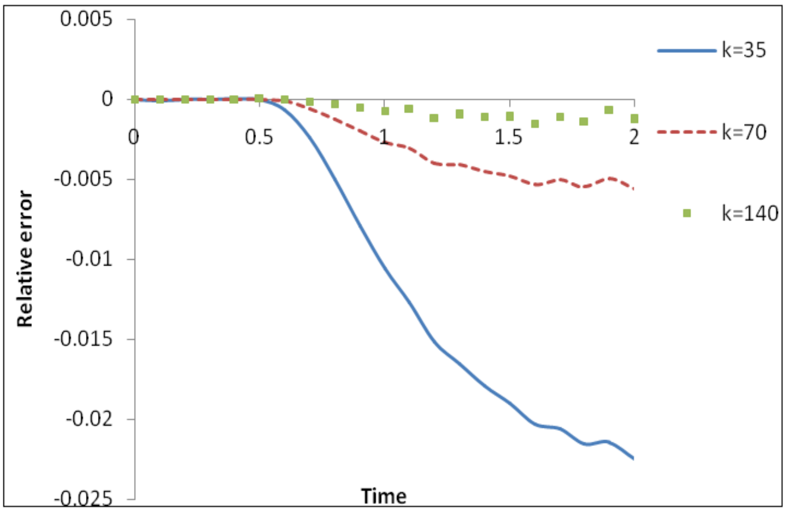

Figure 5: Comparison of Relative Error for Different Number of Elements for P=10

Comparison of OCM and HCM

In Figure 6 the behavior of relative error is shown for P = 10 for orthogonal collocation method (OCM) and HCM. The similar problem is solved by OCM for 5 interior collocation points. In this figure it is clear that relative error is very high for OCM as compared to HCM. As time increases, the behavior of solution profile is very abrupt. In case of HCM, the relative error is very small and is less than 1 % which goes upto more than 200 times in case of OCM. In Table 1 and 2, the analytic and numerical values are compared for P = 6 and P = 10 for OCM and HCM. It is clear from this table that with the increase in time relative error shoots where as in case of HCM, the values approach to steady state condition very smoothly.

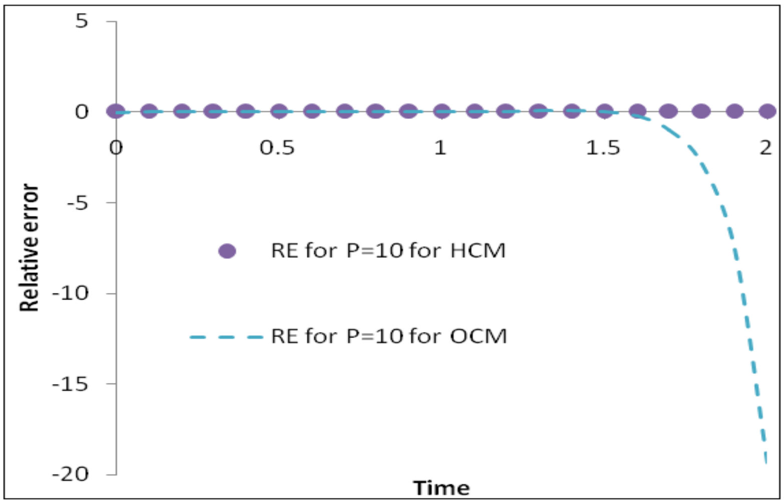

Figure 6: Behavior of Relative Error for OCM and HCM

Table 1: Comparison of Analytic and Numerical Results Calculated by HCM and OCM

t	P=6, Analytic	P=6, HCM	P=6, OCM	RE=6, HCM	RE=6, OCM
0.0	1.000×10^{0}	1.000×10^{0}	1.020×10^{0}	0.000×10^{0}	2.040×10^{-2}
0.1	1.000×10^{0}	1.000×10^{0}	9.974×10^{-1}	0.000×10^{0}	2.590×10^{-3}
0.5	9.915×10^{-1}	9.914×10^{-1}	9.921×10^{-1}	3.025×10^{-5}	6.354×10^{-4}
0.8	7.455×10^{-1}	7.456×10^{-1}	7.374×10^{-1}	1.610×10^{-4}	1.089×10^{-2}
1.2	2.144×10^{-1}	2.145×10^{-1}	2.098×10^{-1}	5.597×10^{-4}	2.164×10^{-2}
1.8	1.168×10^{-2}	1.168×10^{-2}	1.582×10^{-2}	5.993×10^{-4}	3.540×10^{-1}
2.0	3.882×10^{-3}	3.885×10^{-3}	8.783×10^{-3}	7.985×10^{-4}	1.262×10^{0}

Table 2: Comparison of Analytic and Numerical Results Calculated by HCM and OCM

t	P=10, Analytic	P=10, HCM	P=10, OCM	RE=10, HCM	RE=10, OCM
0.0	1.000×10^{0}	1.000×10^{0}	1.026×10^{0}	0.000×10^{0}	2.560×10^{-2}
0.1	1.000×10^{0}	1.000×10^{0}	9.953×10^{-1}	0.000×10^{0}	4.720×10^{-3}
0.5	9.991×10^{-1}	9.991×10^{-1}	1.001×10^{0}	3.002×10^{-5}	1.802×10^{-3}
0.8	8.185×10^{-1}	8.187×10^{-1}	8.021×10^{-1}	3.298×10^{-4}	1.999×10^{-2}
1.1	2.934×10^{-1}	2.935×10^{-1}	2.849×10^{-1}	6.135×10^{-4}	2.904×10^{-2}
1.6	1.162×10^{-2}	1.163×10^{-2}	1.440×10^{-2}	1.549×10^{-3}	2.390×10^{-1}
1.8	2.372×10^{-3}	2.375×10^{-3}	9.168×10^{-3}	1.391×10^{-3}	2.865×10^{0}
2.0	4.382×10^{-4}	4.387×10^{-4}	8.951×10^{-3}	1.279×10^{-3}	1.943×10^{1}

CONCLUSIONS

In present study, the technique of Hermite collocation is presented to solve two point boundary value problems. From Tables 1and Table 2 it is observed that HCM is better than OCM for high range of parameters. It is also observed that HCM gives better results than OCM which converge to steady state condition to the desired accuracy of less than 1% for P varying from 1 to 140 whereas on the other hand OCM gives the results deviated from analytic ones. Hence technique of HCM is found to be better and more convergent than OCM to solve two point boundary value problems.

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